Math Sci (2014) 8:95–108
DOI 10.1007/s40096-014-0134-5

ORIGINAL RESEARCH

Quadruple fixed point theorems for compatible type mappings without mixed g-monotone property in partially ordered b-metric spaces

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Received: 15 September 2014 / Accepted: 12 November 2014 / Published online: 28 November 2014

Abstract In this paper, we establish quadruple fixed point theorems for compatible type mappings in partially ordered b-metric spaces without the mixed g-monotone property under some conditions. Also, an example is given to show our results are real generalization of known results in quadruple fixed point theory.

Keywords Quadruple fixed point · Compatible and w-compatible mappings · Partially ordered b-metric spaces

Mathematics Subject Classification 54H25 · 47H10

Introduction and preliminaries

The concept of b-metric space was introduced and studied by Bakhtin [7] and later used by Czerwik [13, 14] which is a generalization of the usual metric space. After that, several papers have dealt with fixed point theory for single-valued and multi-valued operators in b-metric spaces have been obtained (see, e.g., [2, 5, 11, 12, 26, 28]).

Existence of coupled fixed point was introduced by Guo and Lakshmikantham [16]. In 2006, Gnana-Bhaskar and Lakshmikantham [10] introduced the concept of mixed monotone property in partially ordered metric space. Afterward, Lakshmikantham and Ćirić in [24] extended these results by giving the definition of the g-monotone property. Many papers have been reported on coupled fixed point theory (see, e.g., [1, 3, 4, 9, 18, 27]). In 2011, Vasile Berinde and Marin Borcut [8] extended and generalized the results of [10] and introduced the concept of a tripled fixed point and the mixed monotone property of a mapping $F: X^3 \to X$. For more details on tripled fixed point results, we refer to [6, 26]. Recently, Karapinar and Luong [19] introduced the concept of a quadruple fixed point and the mixed monotone property of a mapping $F: X^4 \to X$ and they presented some new quadruple fixed point results. For a survey of quadruple fixed point theorems and related fixed points we refer the reader to [20–22].

Definition 1.1 [14]. Let X be a nonempty set and $s \geq 1$ a given real number. A function $d: X^2 \to R^+$ is called a b-metric provided that, for all $x, y, z \in X$, the following conditions hold:

(b_1) $d(x,y) = 0$ if and only if $x = y$,
(b_2) $d(x,y) = d(y,x)$,
(b_3) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

The pair (X, d) is called a b-metric space with parameter s.

Remark 1.1 It is obvious that any metric space must be a b-metric space where a b-metric space is a metric space when $s = 1$. The following example show that in general a b-metric need not necessarily be a metric space (see also [29]).

Example 1.1 [2]. Let (X, d) be a metric space and $\rho(x,y) = (d(x,y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$. However, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space. For example, if $X = R$ is the set of real numbers and $d(x,y) = |x - y|$ is the usual Euclidean metric, then $\rho(x,y) = (x-y)^2$ is a b-metric on R with $s = 2$, but is not a metric on R.

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Definition 1.2 [11]. Let (X,d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called

(i) b-convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$,
(ii) b-Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Proposition 1.1 ([11] Remark 2.1) *In a b-metric space the following assertions hold*:

(i) *A b-convergent sequence has a unique limit.*
(ii) *Each b-convergent sequence is b-Cauchy.*
(iii) *In general, a b-metric is not continuous.*

Definition 1.3 [11]. Let (X,d) and $(\bar{X},\bar{d})$ be two b-metric spaces.

(i) The space (X,d) is b-complete if every b-Cauchy sequence in X b-converges.
(ii) A function $f : X \to \bar{X}$ is b-continuous at a point $x \in X$ if it is b-sequentially continuous at x, that is, whenever $\{x_n\}$ is b-convergent to x, $\{f(x_n)\}$ is b-convergent to $f(x)$.

Definition 1.4 [11]. The b-metric space (X,d) is b-complete if every b-Cauchy sequence in X b-converges.

It should be noted that, in general a b-metric function $d(x,y)$ for $s > 1$ is not jointly continuous in all its variables. The following example on a b-metric which is not continuous.

Example 1.2 [17]. Let $X = \mathbf{N} \cup \{\infty\}$ and let $d : X \times X \to R$ be defined by

$$d(m,n) = \begin{cases} 0, & \text{if } m = n, \\ \left|\frac{1}{m} - \frac{1}{n}\right|, & \text{if one of } m,n \text{ is even and the other is even or } \infty, \\ 5, & \text{if one of } m,n \text{ is odd and the other is odd (and } m \neq n) \text{ or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

Then considering all possible cases, it can be checked that for all $m, n, p \in X$, we have

$$d(m,p) \le \frac{5}{2}(d(m,n) + d(n,p)).$$

Thus, (X,d) is a b-metric space (with $s = \frac{5}{3}$). Let $x_n = 2n$ for each $n \in \mathbf{N}$. Then

$$d(2n,\infty) = \frac{1}{2n} \to 0 \quad \text{as } n \to \infty,$$

that is, $x_n \to \infty$, but $d(x_n, 1) = 2 \nrightarrow 5 = d(\infty, 1)$ as $n \to \infty$.

Since, in general, a b-metric is not continuous, we need the following lemma about the b-convergent sequences in the proof of our main result.

Lemma 1.1 [2]. *Let (X,d) be a b-metric space with $s \ge 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ b-converge to x, y, respectively. Then, we have*

$$\frac{1}{s^2} d(x,y) \le \liminf_{n\to\infty} d(x_n, y_n) \le \limsup_{n\to\infty} d(x_n, y_n) \le s^2 d(x,y).$$

In particular, if $x = y$ *then* $\lim_{n\to\infty} d(x_n, y_n) = 0$. *Moreover, for each $z \in X$ we have*

$$\frac{1}{s} d(x,z) \le \liminf_{n\to\infty} d(x_n, z) \le \limsup_{n\to\infty} d(x_n, z) \le s d(x,z).$$

Definition 1.5 Let X be a nonempty set and let $F : X^4 \to X$, $g : X \to X$. An element $(x,y,z,w) \in X^4$ is called

(i) [19] a quadruple fixed point of F if

$$F(x,y,z,w) = x, \qquad F(y,z,w,x) = y,$$
$$F(z,w,x,y) = z \quad \text{and} \quad F(w,x,y,z) = w.$$

(ii) [25] a quadruple coincidence point of F and g if

$$F(x,y,z,w) = gx, \qquad F(y,z,w,x) = gy,$$
$$F(z,w,x,y) = gz \quad \text{and} \quad F(w,x,y,z) = gw.$$

(gx, gy, gz, gw) is said to be a quadruple point of coincidence of F and g.

(iii) [25] a quadruple common fixed point of F and g if

$$F(x,y,z,w) = gx = x, \qquad F(y,z,w,x) = gy = y,$$
$$F(z,w,x,y) = gz = z \quad \text{and} \quad F(w,x,y,z) = gw = w.$$

Definition 1.6 [25]. Let $(X, \preceq)$ be a partially ordered set and let $F : X^4 \to X$, $g : X \to X$. The mapping F is said to have the mixed g-monotone property if for any $x, y, z, w \in X$,

$$x_1, x_2 \in X,\ gx_1 \le gx_2 \quad \text{implies} \quad F(x_1,y,z,w) \le F(x_2,y,z,w),$$
$$y_1, y_2 \in X,\ gy_1 \le gy_2 \quad \text{implies} \quad F(x,y_2,z,w) \le F(x,y_1,z,w),$$
$$z_1, z_2 \in X,\ gz_1 \le gz_2 \quad \text{implies} \quad F(x,y,z_1,w) \le F(x,y,z_2,w),$$
$$\text{and} \quad w_1, w_2 \in X,\ gw_1 \le gw_2 \text{ implies } F(x,y,z,w_2) \le F(x,y,z,w_1).$$

In particular, when $g = i_X$, then from [19] we say that F has the mixed monotone property that is, for any $x, y, z, w \in X$,

$$x_1, x_2 \in X,\ x_1 \le x_2 \quad \text{implies} \quad F(x_1,y,z,w) \le F(x_2,y,z,w),$$
$$y_1, y_2 \in X,\ y_1 \le y_2 \quad \text{implies} \quad F(x,y_2,z,w) \le F(x,y_1,z,w),$$
$$z_1, z_2 \in X,\ z_1 \le z_2 \quad \text{implies} \quad F(x,y,z_1,w) \le F(x,y,z_2,w),$$
$$\text{and} \quad w_1, w_2 \in X,\ w_1 \le w_2 \text{ implies } F(x,y,z,w_2) \le F(x,y,z,w_1).$$

The concept of an altering distance function was introduced by Khan et al. [23] as follows.

Definition 1.7 [23]. A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if

(i) ψ is non-decreasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.8 [1]. The mappings $F : X \times X \to X$ and $f : X \to X$ are called w-compatible if $f(F(x,y)) = F(fx, fy)$ whenever $f(x) = F(x,y)$ and $f(y) = F(y,x)$.

In 2012, Dorić et al. [15] established coupled fixed point results without the mixed monotone property. This property is automatically satisfied in the case of a totally ordered space. Therefore, these results can be applied in a much wider class of problems. Now, we state a property due to Dorić et al. [15].

If elements x, y of a partially ordered set $(X, \preceq)$ are comparable (i.e., $x \preceq y$ or $y \succeq x$ holds) we will write $x \asymp y$. Let $g : X \to X$ and $F : X \times X \to X$. We will consider the following condition:

$$\text{if } x,y,u,v \in X \text{ are such that } gx \asymp F(x,y) = gu \text{ then } F(x,y) \asymp F(u,v). \tag{1.1}$$

In particular, when $g = i_X$, it reduces to

$$\text{for all } x,y,v \text{ if } x \asymp F(x,y) \text{ then } F(x,y) \asymp F(F(x,y),v). \tag{1.2}$$

The aim of this paper is to extend the property due to Dorić et al. [15] to the case of mappings $g : X \to X$, $F : X^4 \to X$, and show that a mixed monotone property in quadruple fixed point results for mappings in partially ordered b-metric spaces can be replaced by another property which is often easy to check in the case of a totally ordered space. We prove the existence of quadruple coincidence and uniqueness quadruple common fixed point theorems for a compatible and w-compatible mappings satisfying generalized contraction in partially ordered b-metric spaces without the mixed g-monotone property. Also, we state an example showing that our results are effective.

Quadruple coincidence point theorems

Let $g : X \to X$ and $F : X^4 \to X$. We consider the following condition:

$$\text{if } x,y,z,w,u,v,r,t \in X \text{ are such that } gx \asymp F(x,y,z,w) = gu \text{ then } F(x,y,z,w) \asymp F(u,v,r,t). \tag{2.1}$$

In particular, when $g = i_X$, it reduces to

$$\text{for all } x,y,v,r,t \text{ if } x \asymp F(x,y,z,w) \text{ then } F(x,y,z,w) \asymp F(F(x,y,z,w),v,r,t). \tag{2.2}$$

We will show by examples the condition (2.1), ((2.2) resp.) may be satisfied when F does not have the mixed g-monotone property, (monotone property resp.).

Example 2.1 Let $X = \{a,b,c,d\}$, $\preceq = \{(a,a),(b,b),(c,c),(d,d),(a,b),(c,d)\}$,

$$g : \begin{pmatrix} a & b & c & d \\ c & d & c & d \end{pmatrix},$$

$$F : \begin{pmatrix} (a,y,z,w) & (b,y,z,w) & (c,y,z,w) & (d,y,z,w) \\ b & a & c & d \end{pmatrix},$$

for all $y,z,w \in X$. Since $ga = c \preceq gb = d$ but $F(a,y,z,w) \succeq F(b,y,z,w)$ for all $y,z,w \in X$, the mapping F does not have the mixed g-monotone property. But it has property (2.1) where

(i) For each $y,z,w \in X$, we get $gc \asymp F(c,y,z,w)$ and $F(c,y,z,w) \asymp F(c,v,r,t)$ for all $v,r,t \in X$.
(ii) For each $y,z,w \in X$, we get $gd \asymp F(d,y,z,w)$ and $F(d,y,z,w) \asymp F(d,v,r,t)$ for all $v,r,t \in X$.
(iii) The other two cases are trivial.

Example 2.2 Let $X = \{a,b,c,d\}$, $\preceq = \{(a,a),(b,b),(c,c),(d,d),(a,b),(c,d)\}$,

$$F : \begin{pmatrix} (a,y,z,w) & (b,y,z,w) & (c,y,z,w) & (d,y,z,w) \\ b & a & c & d \end{pmatrix}$$

for all $y,z,w \in X$. Since $a \preceq b$ but $F(a,y,z,w) = b \succeq a = F(b,y,z,w)$ for all $y,z,w \in X$, the mapping F does not have the mixed monotone property. But it has property (2.2) since

(i) For each $y,z,w \in X$, we get $a \asymp F(a,y,z,w)$ and $F(a,y,z,w) = b \asymp a = F(F(a,y,z,w),v,r,t)$ for all $v,r,t \in X$.
(ii) For each $y,z,w \in X$, we get $b \asymp F(b,y,z,w)$ and $F(b,y,z,w) = a \asymp b = F(F(b,y,z,w),v,r,t)$ for all $v,r,t \in X$.
(iii) The other two cases are trivial.

Definition 2.1 Let (X, d) be a b-metric space and let $g: X \to X$, $F: X^4 \to X$. The mappings g and F are said to be compatible if

$$\lim_{n\to\infty} d(gF(x_n, y_n, z_n, w_n), F(gx_n, gy_n, gz_n, gw_n)) = 0,$$
$$\lim_{n\to\infty} d(gF(y_n, z_n, w_n, x_n), F(gy_n, gz_n, gw_n, gx_n)) = 0,$$
$$\lim_{n\to\infty} d(gF(z_n, w_n, x_n, y_n), F(gz_n, gw_n, gx_n, gy_n)) = 0$$
$$\text{and} \quad \lim_{n\to\infty} d(gF(w_n, x_n, y_n, z_n), F(gw_n, gx_n, gy_n, gz_n)) = 0,$$

hold whenever $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ are sequences in X such that

$$\lim_{n\to\infty} F(x_n, y_n, z_n, w_n) = \lim_{n\to\infty} gx_n,$$
$$\lim_{n\to\infty} F(y_n, z_n, w_n, x_n) = \lim_{n\to\infty} gy_n,$$
$$\lim_{n\to\infty} F(z_n, w_n, x_n, y_n) = \lim_{n\to\infty} gz_n \quad \text{and}$$
$$\lim_{n\to\infty} F(w_n, x_n, y_n, z_n) = \lim_{n\to\infty} gw_n.$$

Definition 2.2 The mappings $F: X^4 \to X$ and $g: X \to X$ are called w-compatible if $g(F(x, y, z, w)) = F(gx, gy, gz, gw)$ whenever $gx = F(x, y, z, w)$, $gy = F(y, z, w, x)$, $gz = F(z, w, x, y)$ and $gw = F(w, x, y, z)$.

Remark 2.1 In an altering distance function $\psi: [0, \infty) \to [0, \infty)$, since ψ is non-decreasing then for any $a, b, c, d \in [0, \infty)$ the following holds.

$$\psi(\max\{a, b, c, d\}) = \max\{\psi(a), \psi(b), \psi(c), \psi(d)\}.$$

The triple $(X, d, \preceq)$ is called a partially ordered b-metric space if $(X, \preceq)$ is a partially ordered set and (X, d) is a b-metric space.

Our first result is the following.

Theorem 2.1 *Let $(X, d, \preceq)$ be a partially ordered complete b-metric space with parameter $s \geq 1$. Let $F: X^4 \to X$ and $g: X \to X$ be two mappings such that the following hold:*

(i) *g and F are b-continuous,*
(ii) *$F(X^4) \subseteq g(X)$, g and F are compatible,*
(iii) *g and F satisfy property (2.1),*
(iv) there exist *$x_0, y_0, z_0, w_0 \in X$ such that $gx_0 \asymp F(x_0, y_0, z_0, w_0)$,*
$gy_0 \asymp F(y_0, z_0, w_0, x_0)$, $gz_0 \asymp F(z_0, w_0, x_0, y_0)$ and $gw_0 \asymp F(w_0, x_0, y_0, z_0)$,
(v) *there exist an altering distance function ψ and $\phi: [0, \infty)^4 \to [0, \infty)$ is continuous with $\phi(t_1, t_2, t_3, t_4) = 0$ if and only if $t_1 = t_2 = t_3 = t_4 = 0$ such that*

$$\begin{aligned}&\psi(sd(F(x, y, z, w), F(u, v, r, t)))\\ &\leq \psi(\max\{d(gx, gu), d(gy, gv), d(gz, gr), d(gw, gt)\})\\ &- \phi(d(gx, gu), d(gy, gv), d(gz, gr), d(gw, gt)),\end{aligned} \tag{2.3}$$

for all $x, y, z, w, u, v, r, t \in X$ and $gx \asymp gu$, $gy \asymp gv$, $gz \asymp gr$ and $gw \asymp gt$.

Then, F and g have a quadruple coincidence point.

Proof Let $x_0, y_0, z_0, w_0 \in X$ be such that condition (iv) holds. Since $F(X^4) \subseteq g(X)$, then we can choose $x_1, y_1, z_1, w_1 \in X$ such that

$$\begin{aligned}&gx_1 = F(x_0, y_0, z_0, w_0), \quad gy_1 = F(y_0, z_0, w_0, x_0),\\ &gz_1 = F(z_0, w_0, x_0, y_0) \quad \text{and} \quad gw_1 = F(w_0, x_0, y_0, z_0).\end{aligned} \tag{2.4}$$

By continuing this process, we can construct sequences $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{w_n\}$ in X such that

$$\begin{aligned}&gx_{n+1} = F(x_n, y_n, z_n, w_n),\ gy_{n+1} = F(y_n, z_n, w_n, x_n),\\ &gz_{n+1} = F(z_n, w_n, x_n, y_n),\ \text{and}\ gw_{n+1} = F(w_n, x_n, y_n, z_n)\\ &\text{for all}\ n \geq 0.\end{aligned} \tag{2.5}$$

We show that

$$\begin{aligned}&gx_n \asymp gx_{n+1}, \quad gy_n \asymp gy_{n+1}, \quad gz_n \asymp gz_{n+1}\\ &\text{and} \quad gw_n \asymp gw_{n+1} \quad \text{for } n \geq 0.\end{aligned} \tag{2.6}$$

So, we use the mathematical induction. By condition (iv) and using (2.4) we get $gx_0 \asymp gx_1$, $gy_0 \asymp gy_1$,$gz_0 \asymp gz_1$, and $gw_0 \asymp gw_1$. So (2.6) holds for $n = 0$. We assume that (2.6) holds for some $n > 0$, that is $gx_n \asymp gx_{n+1}$, $gy_n \asymp gy_{n+1}$, $gz_n \asymp gz_{n+1}$, and $gw_n \asymp gw_{n+1}$, we get

$$gx_n = F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}) \asymp F(x_n, y_n, z_n, w_n) = gx_{n+1},$$

$$gy_n = F(y_{n-1}, z_{n-1}, w_{n-1}, x_{n-1}) \asymp F(y_n, z_n, w_n, x_n) = gy_{n+1},$$

$$gz_n = F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}) \asymp F(z_n, w_n, x_n, y_n) = gz_{n+1},$$

$$gw_n = F(w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}) \asymp F(w_n, x_n, y_n, z_n) = gw_{n+1}.$$

Hence from condition (iii) we conclude that

$$F(x_n, y_n, z_n, w_n) \asymp F(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1}),$$
$$F(y_n, z_n, w_n, x_n) \asymp F(y_{n+1}, z_{n+1}, w_{n+1}, x_{n+1}),$$
$$F(z_n, w_n, x_n, y_n) \asymp F(z_{n+1}, w_{n+1}, x_{n+1}, y_{n+1}),$$
$$F(w_n, x_n, y_n, z_n) \asymp F(w_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}).$$

So, $gx_{n+1} \asymp gx_{n+2}$, $gy_{n+1} \asymp gy_{n+2}$,$gz_{n+1} \asymp gz_{n+2}$ and $gw_{n+1} \asymp gw_{n+2}$. Thus (2.6) holds for all $n \in \mathbf{N}$. Suppose that for some $k \in \mathbf{N}$,

$$gx_k = gx_{k+1}, \quad gy_k = gy_{k+1},$$
$$gz_k = gz_{k+1} \quad \text{and} \quad gw_k = gw_{k+1},$$

then by (2.5) we get $gx_k = F(x_k, y_k, z_k, w_k)$, $gy_k = F(y_k, z_k, w_k, x_k)$, $gz_k = F(z_k, w_k, x_k, y_k)$ and $gw_k = F(w_k, x_k, y_k, z_k)$. Hence (x_k, y_k, z_k, w_k) is a quadruple coincidence point of F and g. So, we assume that for all $n \in \mathbf{N}$ at least $gx_n \neq gx_{n+1}$ or $gy_n \neq gy_{n+1}$ or $gz_n \neq gz_{n+1}$ or $gw_n \neq gw$ $n+1$. Since $gx_n \asymp gx_{n+1}$, $gy_n \asymp gy_{n+1}$, $gz_n \asymp gz_{n+1}$ and $gw_n \asymp gw_{n+1}$ for all $n \in \mathbf{N}$, then from (2.3) and (2.5) we obtain

$$\psi(sd(gx_n, gx_{n+1}) = \psi(sd(F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}), F(x_n, y_n, z_n, w_n))) \leq \psi(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n)\}) - \phi(d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n)), \tag{2.7}$$

$$\psi(sd(gy_n, gy_{n+1})) = \psi(sd(F(y_{n-1}, z_{n-1}, w_{n-1}, x_{n-1}), F(y_n, z_n, w_n, x_n))) \leq \psi(\max\{d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n)\}) - \phi(d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n)), \tag{2.8}$$

$$\psi(sd(gz_n, gz_{n+1})) = \psi(sd(F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}), F(z_n, w_n, x_n, y_n))) \leq \psi(\max\{d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}) - \phi(d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)), \tag{2.9}$$

and

$$\psi(sd(gw_n, gw_{n+1})) = \psi(sd(F(w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}), F(w_n, x_n, y_n, z_n))) \leq \psi(\max\{d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\}) - \phi(d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)). \tag{2.10}$$

Set

$$\delta_n = \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1}), d(gw_n, gw_{n+1})\}.$$

From (2.7) to (2.10) and Remark 2.1, it follows that

$$\psi(s\delta_n) = \max\{\psi(sd(gx_n, gx_{n+1})), \psi(sd(gy_n, gy_{n+1})), \psi(sd(gz_n, gz_{n+1})), \psi(sd(gw_n, gw_{n+1}))\} \leq \psi(\delta_{n-1}) - \min\{\phi(d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n)), \phi(d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n)), \phi(d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)), \phi(d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n))\}. \tag{2.11}$$

and since ψ is non-decreasing then from (2.11) we have that

$$\psi(\delta_n) \leq \psi(s\delta_n) \leq \psi(\delta_{n-1}) - \min\{\phi(d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n)), \phi(d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n)), \phi(d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)), ; \phi(d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n))\}. \tag{2.12}$$

Hence,

$$\psi(\delta_n) \leq \psi(\delta_{n-1}) \text{ for all } n \in \mathbf{N}. \tag{2.13}$$

Since ψ is non-decreasing, we have $\delta_n \leq \delta_{n-1}$ for all n. Therefore, $\{\delta_n\}$ is a non-increasing sequence, so there is some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \delta.$$

Letting $n \to \infty$, in (2.12), we get

$$\psi(\delta) \leq \psi(\delta) - \min\{\lim_{n\to\infty} \phi(d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n)), \lim_{n\to\infty} \phi(d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n)), \lim_{n\to\infty} \phi(d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)), \lim_{n\to\infty} \phi(d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n))\} \leq \psi(\delta).$$

Hence,

$$\min \lim_{n\to\infty} \phi(d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n)), \lim_{n\to\infty} \phi(d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n)), \lim_{n\to\infty} \phi(d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)), \lim_{n\to\infty} \phi(d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n))\} = 0.$$

That is

$$\lim_{n\to\infty} \phi(d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n)) = 0 \text{ or } \lim_{n\to\infty} \phi(d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n)) = 0 \text{ or } \lim_{n\to\infty} \phi(d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)) = 0 \quad \text{or} \quad \lim_{n\to\infty} \phi(d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)) = 0.$$

So, using the properties of ϕ, we have

$$\lim_{n\to\infty} d(gx_{n-1}, gx_n) = 0, \qquad \lim_{n\to\infty} d(gy_{n-1}, gy_n) = 0,$$
$$\lim_{n\to\infty} d(gz_{n-1}, gz_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(gw_{n-1}, gw_n) = 0.$$

Thus, $\lim_{n\to\infty} \delta_{n-1} = 0$. Therefore $\delta = 0$. Now, we show that $\{gx_n\}, \{gy_n\}, \{gz_n\}$, and $\{gw_n\}$ are b-Cauchy sequences in (X, d), that is, we show that for every $\varepsilon > 0$, there exists $k \in \mathbf{N}$ such that for all $m, n \geq k$,

$$\max\{d(gx_m, gx_n), d(gy_m, gy_n), d(gz_m, gz_n), d(gw_m, gw_n)\} < \varepsilon.$$

Suppose the contrary, that is at least one of the sequences $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$ and $\{gw_n\}$ is not a b-Cauchy sequence, so there exists $\varepsilon > 0$ for which we can find subsequences $\{gx_{m(k)}\}$, $\{gx_{n(k)}\}$ of $\{gx_n\}$, $\{gy_{m(k)}\}, \{gy_{n(k)}\}$ of $\{gy_n\}$,$\{gz_{m(k)}\}$, $\{gz_{n(k)}\}$ of $\{gz_n\}$ and $\{gw_{m(k)}\}$, $\{gw_{n(k)}\}$ of $\{gw_n\}$ with $n(k) > m(k) \geq k$ such that

$$\max\{d(gx_{m(k)}, gx_{n(k)}), d(gy_{m(k)}, gy_{n(k)}), d(gz_{m(k)}, gz_{n(k)}), d(gw_{m(k)}, gw_{n(k)})\} \geq \varepsilon, \tag{2.14}$$

for every integer k, let $n(k)$ be the least positive integer with $n(k) > m(k) \geq k$ satisfying (2.14) and such that

$$\max\{d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1}), d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1})\} < \varepsilon. \tag{2.15}$$

Using the b-triangle inequality, we get

$$d(gx_{m(k)}, gx_{n(k)}) \leq s[d(gx_{m(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{n(k)})] d(gy_{m(k)}, gy_{n(k)}) \leq s[d(gy_{m(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{n(k)})] d(gz_{m(k)}, gz_{n(k)}) \leq s[d(gz_{m(k)}, gz_{n(k)-1}) + d(gz_{n(k)-1}, gz_{n(k)})] \text{ and } d(gw_{m(k)}, gw_{n(k)}) \leq s[d(gw_{m(k)}, gw_{n(k)-1}) + d(gw_{n(k)-1}, gw_{n(k)})]. \tag{2.16}$$

Hence from (2.14) and (2.16), we have

$$\begin{aligned}\varepsilon &\leq \max\{d(gx_{m(k)}, gx_{n(k)}), d(gy_{m(k)}, gy_{n(k)}), d(gz_{m(k)}, gz_{n(k)}), d(gw_{m(k)}, gw_{n(k)})\} \leq s[\max\{d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1}), d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1})\} \\ &\quad + \max\{d(gx_{n(k)-1}, gx_{n(k)}), d(gy_{n(k)-1}, gy_{n(k)}), d(gz_{n(k)-1}, gz_{n(k)}), d(gw_{n(k)-1}, gw_{n(k)})\}] = s\max\{d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1}), d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1})\} + s\delta_{n(k)-1}.\end{aligned}$$

Taking the upper and lower limits as $k \to \infty$ in the above inequality, from (2.14), (2.15) and as $\lim_{n\to\infty} \delta_{n-1} = 0$, we conclude that

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} \max\{d(gx_{m(k)}, gx_{n(k)}), d(gy_{m(k)}, gy_{n(k)}), d(gz_{m(k)}, gz_{n(k)}), d(gw_{m(k)}, gw_{n(k)})\} < \varepsilon, \tag{2.17}$$

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} \max\{d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1}), d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1})\} < \varepsilon, \tag{2.18}$$

and

$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} \max\{d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1}), d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1})\} < \varepsilon. \tag{2.19}$$

Also, from the b-triangle inequality we obtain

$$\begin{aligned}\varepsilon &\leq \max\{d(gx_{m(k)}, gx_{n(k)}), d(gy_{m(k)}, gy_{n(k)}), d(gz_{m(k)}, gz_{n(k)}), d(gw_{m(k)}, gw_{n(k)})\} \leq s[\max\{d(gx_{m(k)}, gx_{m(k)+1}), d(gy_{m(k)}, gy_{m(k)+1}), d(gz_{m(k)}, gz_{m(k)+1}), d(gw_{m(k)}, gw_{m(k)+1})\} \\ &\quad + \max\{d(gx_{m(k)+1}, gx_{n(k)}), d(gy_{m(k)+1}, gy_{n(k)}), d(gz_{m(k)+1}, gz_{n(k)}), d(gw_{m(k)+1}, gw_{n(k)})\}] \\ &= s\delta_{m(k)} + s\max\{d(gx_{m(k)+1}, gx_{n(k)}), d(gy_{m(k)+1}, gy_{n(k)}), d(gz_{m(k)+1}, gz_{n(k)}), d(gw_{m(k)+1}, gw_{n(k)})\},\end{aligned}$$

and

$$\begin{aligned}&\max\{d(gx_{m(k)+1}, gx_{n(k)}), d(gy_{m(k)+1}, gy_{n(k)}), d(gz_{m(k)+1}, gz_{n(k)}), d(gw_{m(k)+1}, gw_{n(k)})\} \leq s[\max\{d(gx_{m(k)+1}, gx_{m(k)}) + gy_{m(k)+1}, d(gy_{m(k)}) + d(gz_{m(k)+1}, gz_{m(k)}) + d(gw_{m(k)+1}, gw_{m(k)})\} \\ &\quad + \max\{d(gx_{m(k)}, gx_{n(k)}), d(gy_{m(k)}, gy_{n(k)}), d(gz_{m(k)}, gz_{n(k)}), d(gw_{m(k)}, gw_{n(k)})\}] = s\delta_{m(k)} + s\max\{d(gx_{m(k)}, gx_{n(k)}), d(gy_{m(k)}, gy_{n(k)}), d(gz_{m(k)}, gz_{n(k)}), d(gw_{m(k)}, gw_{n(k)})\}.\end{aligned}$$

Taking the upper limit as $k \to \infty$ in the above two inequalities, using (2.14) and as $\lim_{n\to\infty} \delta_{n-1} = 0$, we have

$$\frac{\varepsilon}{s} \leq \limsup_{n\to\infty} \max\{d(gx_{m(k)+1}, gx_{n(k)}), d(gy_{m(k)+1}, gy_{n(k)}), d(gz_{m(k)+1}, gz_{n(k)}), d(gw_{m(k)+1}, gw_{n(k)})\} < s\varepsilon. \tag{2.20}$$

Since $gx_n \asymp gx_{n+1}$, $gy_n \asymp gy_{n+1}$, $gz_n \asymp gz_{n+1}$ and $gw_n \asymp gw_{n+1}$ for all $n \geq 0$, then $gx_{m(k)} \asymp gx_{n(k)-1}$, $gy_{m(k)} \asymp gy\, n(k) - 1$,$gz_{m(k)} \asymp gz_{n(k)-1}$ and $gw_{m(k)} \asymp gw_{n(k)-1}$.

Putting $x = x_{m(k)}$, $y = y_{m(k)}$, $z = z_{m(k)}$, $w = w_{m(k)}$, $u = x_{n(k)-1}$, $v = y_{n(k)-1}$, $r = z_{n(k)-1}$, $t = w_{n(k)-1}$, in (2.3) for all $k \geq 0$, we conclude that

$$
\begin{aligned}
&\psi(sd(gx_{m(k)+1}, gx_{n(k)})) \\
&= \psi(sd(F(x_{m(k)}, y_{m(k)}, z_{m(k)}, w_{m(k)}), F(x_{n(k)-1}, y_{n(k)-1}, \\
&\quad z_{n(k)-1}, w_{n(k)-1}))) \\
&\leq \psi(\max\{d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1}), \\
&\quad d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1})\}) \\
&- \phi(d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1}), \\
&\quad d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1})),
\end{aligned} \tag{2.21}
$$

$$
\begin{aligned}
&\psi(sd(gy_{m(k)+1}, gy_{n(k)})) \\
&= \psi(sd(F(y_{m(k)}, z_{m(k)}, w_{m(k)}, x_{m(k)}), F(y_{n(k)-1}, \\
&\quad z_{n(k)-1}, w_{n(k)-1}, x_{n(k)-1}))) \\
&\leq \psi(\max\{d(gy_{m(k)}, gy_{n(k)-1}), d(gz_{m(k)}, gz_{n(k)-1}), \\
&\quad d(gw_{m(k)}, gw_{n(k)-1}), d(gx_{m(k)}, gx_{n(k)-1})\}) \\
&- \phi(d(gy_{m(k)}, gy_{n(k)-1}), d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, \\
&\quad gw_{n(k)-1}), d(gx_{m(k)}, gx_{n(k)-1})),
\end{aligned} \tag{2.22}
$$

$$
\begin{aligned}
&\psi(sd(gz_{m(k)+1}, gz_{n(k)})) \\
&= \psi(sd(F(z_{m(k)}, w_{m(k)}, x_{m(k)}, y_{m(k)}), F(z_{n(k)-1}, \\
&\quad w_{n(k)-1}, x_{n(k)-1}, y_{n(k)-1}))) \\
&\leq \psi(\max\{d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1}), \\
&\quad d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1})\}) \\
&- \phi(d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1}), \\
&\quad d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1})),
\end{aligned} \tag{2.23}
$$

and

$$
\begin{aligned}
&\psi(sd(gw_{m(k)+1}, gw_{n(k)})) \\
&= \psi(sd(F(w_{m(k)}, x_{m(k)}, y_{m(k)}, z_{m(k)}), F(w_{n(k)-1}, x_{n(k)-1}, \\
&\quad y_{n(k)-1}, z_{n(k)-1}))) \\
&\leq \psi(\max\{d(gw_{m(k)}, gw_{n(k)-1}), d(gx_{m(k)}, gx_{n(k)-1}), \\
&\quad d(gy_{m(k)}, gy_{n(k)-1}), d(gz_{m(k)}, gz_{n(k)-1})\}) \\
&- \phi(d(gw_{m(k)}, gw_{n(k)-1}), d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, \\
&\quad gy_{n(k)-1}), d(gz_{m(k)}, gz_{n(k)-1})).
\end{aligned} \tag{2.24}
$$

From (2.21)–(2.24) and Remark 2.1, it follows that

$$
\begin{aligned}
&\psi(s\max\{d(gx_{m(k)+1}, gx_{n(k)}), d(gy_{m(k)+1}, gy_{n(k)}), \\
&\quad d(gz_{m(k)+1}, gz_{n(k)}), d(gw_{m(k)+1}, gw_{n(k)})\}) \\
&\quad \leq \psi(\max\{d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1}), \\
&\quad d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1})\}) \\
&\quad - \min\{\phi(d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1}), d(gz_{m(k)}, \\
&\quad gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1})), \\
&\quad \phi(d(gy_{m(k)}, gy_{n(k)-1}), d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, \\
&\quad gw_{n(k)-1}), d(gx_{m(k)}, gx_{n(k)-1})), \\
&\quad \phi(d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1}), d(gx_{m(k)}, \\
&\quad gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1})), \\
&\quad \phi(d(gw_{m(k)}, gw_{n(k)-1}), d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1}), \\
&\quad d(gz_{m(k)}, gz_{n(k)-1}))\}.
\end{aligned}
$$

Taking the upper limit as $k \to \infty$ in the above inequality and using (2.18) and (2.20), we have

$$
\begin{aligned}
&\psi(\varepsilon) = \psi\left(s\frac{\varepsilon}{s}\right) \\
&\leq \psi(\varepsilon) - \liminf_{k\to\infty} \min\{\phi(d(gx_{m(k)}, gx_{n(k)-1}), \\
&\quad d(gy_{m(k)}, gy_{n(k)-1}), d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1})), \\
&\phi(d(gy_{m(k)}, gy_{n(k)-1}), d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, \\
&\quad gw_{n(k)-1}), d(gx_{m(k)}, gx_{n(k)-1})), \\
&\phi(d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1}), d(gx_{m(k)}, \\
&\quad gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1})), \\
&\phi(d(gw_{m(k)}, gw_{n(k)-1}), d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, \\
&\quad gy_{n(k)-1}), d(gz_{m(k)}, gz_{n(k)-1}))\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\liminf_{k\to\infty} \min\{\phi(d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1}), \\
&\quad d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1})), \\
&\phi(d(gy_{m(k)}, gy_{n(k)-1}), d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, \\
&\quad gw_{n(k)-1}), d(gx_{m(k)}, gx_{n(k)-1})), \\
&\phi(d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1}), \\
&\quad d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1})), \\
&\phi(d(gw_{m(k)}, gw_{n(k)-1}), d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, \\
&\quad gy_{n(k)-1}), d(gz_{m(k)}, gz_{n(k)-1}))\} = 0,
\end{aligned}
$$

Therefore,

$$\liminf_{k\to\infty} \phi(d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1}), d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1}))$$
$$\liminf_{k\to\infty} \phi(d(gy_{m(k)}, gy_{n(k)-1}), d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1}), d(gx_{m(k)}, gx_{n(k)-1}))$$
$$\liminf_{k\to\infty} \phi(d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1}), d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1}))$$
$$\liminf_{k\to\infty} \phi(d(gw_{m(k)}, gw_{n(k)-1}), d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1}), d(gz_{m(k)}, gz_{n(k)-1}))$$

Using the properties of ϕ, it follows that

$$\liminf_{k\to\infty} d(gx_{m(k)}, gx_{n(k)-1}) = 0,$$
$$\liminf_{k\to\infty} d(gy_{m(k)}, gy_{n(k)-1}) = 0,$$
$$\liminf_{k\to\infty} d(gz_{m(k)}, gz_{n(k)-1}) = 0 \quad \text{and}$$
$$\liminf_{k\to\infty} d(gw_{m(k)}, gw_{n(k)-1}) = 0.$$

Hence,

$$\liminf_{k\to\infty} \max\{d(gx_{m(k)}, gx_{n(k)-1}), d(gy_{m(k)}, gy_{n(k)-1}), d(gz_{m(k)}, gz_{n(k)-1}), d(gw_{m(k)}, gw_{n(k)-1})\} = 0,$$

which is a contradiction to (2.19). Thus, $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$, and $\{gw_n\}$ are b-Cauchy sequences in X. Now, we show that F and g have a quadruple coincidence point. Since X is b-complete and $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$, and $\{gw_n\}$ are b-Cauchy sequences in X, there exists $x, y, z, w \in X$ such that

$$\lim_{n\to\infty} d(gx_n, x) = 0, \quad \lim_{n\to\infty} d(gy_n, y) = 0, \quad \lim_{n\to\infty} d(gz_n, z) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(gw_n, w) = 0. \tag{2.25}$$

From (2.5) and (2.25), we get

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} F(x_n, y_n, z_n, w_n) = x,$$
$$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} F(y_n, z_n, w_n, x_n) = y,$$
$$\lim_{n\to\infty} gz_n = \lim_{n\to\infty} F(z_n, w_n, x_n, y_n) = z$$
$$\lim_{n\to\infty} gw_n = \lim_{n\to\infty} F(w_n, x_n, y_n, z_n) = w.$$

Hence from the compatibility of F and g, we obtain

$$\lim_{n\to\infty} d(gF(x_n, y_n, z_n, w_n), F(gx_n, gy_n, gz_n, gw_n)) = 0,$$
$$\lim_{n\to\infty} d(gF(y_n, z_n, w_n, x_n), F(gy_n, gz_n, gw_n, gx_n)) = 0,$$
$$\lim_{n\to\infty} d(gF(z_n, w_n, x_n, y_n), F(gz_n, gw_n, gx_n, gy_n)) = 0,$$
$$\text{and} \quad \lim_{n\to\infty} d(gF(w_n, x_n, y_n, z_n), F(gw_n, gx_n, gy_n, gz_n)) = 0. \tag{2.26}$$

Further, from the continuity of F and g we get

$$\lim_{n\to\infty} gF(x_n, y_n, z_n, w_n) = \lim_{n\to\infty} ggx_{n+1} = gx,$$
$$\lim_{n\to\infty} F(gx_n, gy_n, gz_n, gw_n) = F(x, y, z, w),$$
$$\lim_{n\to\infty} gF(y_n, z_n, w_n, x_n) = \lim_{n\to\infty} ggy_{n+1} = gy,$$
$$\lim_{n\to\infty} F(gy_n, gz_n, gw_n, gx_n) = F(y, z, w, x),$$
$$\lim_{n\to\infty} gF(z_n, w_n, x_n, y_n) = \lim_{n\to\infty} ggz_{n+1} = gz,$$
$$\lim_{n\to\infty} F(gz_n, gw_n, gx_n, gy_n) = F(z, w, x, y),$$
$$\lim_{n\to\infty} gF(w_n, x_n, y_n, z_n) = \lim_{n\to\infty} ggw_{n+1} = gw \quad \text{and}$$
$$\lim_{n\to\infty} F(gw_n, gx_n, gy_n, gz_n) = F(w, x, y, z).$$

Thus from (2.26) and using Lemma 1.1, we have that $gx = F(x, y, z, w)$, $gy = F(y, z, w, x)$, $gz = F(z, w, x, y)$, $gw = F(w, x, y, z)$. Hence, (x, y, z, w) is a quadruple coincidence point of F and g. □

By removing the continuity and compatibility assumptions of F and g in Theorem 2.1, we prove the following theorem.

Theorem 2.2 *Let $(X, d, \preceq)$ be a partially ordered b-metric space with parameter $s \geq 1$. Let $F : X^4 \to X$ and $g : X \to X$ be two mappings satisfying* (2.3) *for all $x, y, z, w, u, v, r, t \in X$, such that $gx \asymp gu$, $gy \asymp gv$, $gz \asymp gr$ and $gw \asymp gt$, where ψ and ϕ are the same as in Theorem 2.1. Suppose that*

(i) *$F(X^4) \subseteq g(X)$,*
(ii) *F and g satisfy property* (2.1),
(iii) *there exist $x_0, y_0, z_0, w_0 \in X$ such that $gx_0 \asymp F(x_0, y_0, z_0, w_0)$, $gy_0 \asymp F(y_0, z_0, w_0, x_0)$, $gz_0 \asymp F(z_0, w_0, x_0, y_0)$ and $gw_0 \asymp F(w_0, x_0, y_0, z_0)$,*
(iv) *$g(X)$ is a b-complete subspace of X,*
(v) *if $x_n \to x$ when $n \to \infty$ in X, then $x_n \asymp x$ for n sufficiently large.*

Then there exist $x, y, z, w \in X$ *such that* $gx = F(x, y, z, w)$, $gy = F(y, z, w, x)$, $gz = F(z, w, x, y)$, *and* $gw = F(w, x, y, z)$. *Moreover, if* gx_0, gy_0, gz_0 *and* gw_0 *are comparable, then* $gx = gy = gz = gw$, *and if* F *and* g *are* w-*compatible, then* F *and* g *have a quadruple coincidence point of the form* (p, p, p, p).

Proof From Theorem 2.1, we have that $\{gx_n\}, \{gy_n\}, \{gz_n\}$ and $\{gw_n\}$ are b-Cauchy sequences in X. Since $g(X)$ is a b-complete subspace of X and $\{gx_n\}, \{gy_n\}, \{gz_n\}, \{gw_n\} \subseteq g(X)$, there exist $x, y, z, w \in X$ such that

$$\lim_{n\to\infty} d(gx_n, gx) = \lim_{n\to\infty} d(gy_n, gy) = \lim_{n\to\infty} d(gz_n, gz) = \lim_{n\to\infty} d(gw_n, gw) = 0.$$

Since $gx_n \to gx$ when $n \to \infty$ in X, then from condition (v) we obtain $gx_n \asymp gx$ for n sufficiently large. Similarly, we may show that $gy_n \asymp gy$, $gz_n \asymp gz$ and $gw_n \asymp gw$ for n sufficiently large. For such n, using (2.3) we get

$$\begin{aligned}\psi(sd(F(x,y,z,w), gx_{n+1})) &= \psi(sd(F(x,y,z,w), F(x_n, y_n, z_n, w_n)))\\ &\le \psi(\max\{d(gx, gx_n), d(gy, gy_n), d(gz, gz_n), d(gw, gw_n)\})\\ &\quad - \phi(d(gx, gx_n), d(gy, gy_n), d(gz, gz_n), d(gw, gw_n)).\end{aligned}$$

From the above inequality, using Lemma 1.1, as $n \to \infty$, and using the properties of ϕ we obtain

$$\begin{aligned}&\psi\left(\frac{1}{s} d(F(x,y,z,w), gx)\right) \le \psi(s \limsup_{n\to\infty} d(F(x,y,z,w), gx_{n+1}))\\ &= \limsup_{n\to\infty} \psi(sd(F(x,y,z,w), gx_{n+1}))\\ &\le \limsup_{n\to\infty} \psi(\max\{d(gx, gx_n), d(gy, gy_n), d(gz, gz_n), d(gw, gw_n)\})\\ &\quad - \liminf_{n\to\infty} \phi(d(gx, gx_n), d(gy, gy_n), d(gz, gz_n), d(gw, gw_n))\\ &\le \psi(0) - \phi(0,0,0,0) = 0.\end{aligned}$$

Thus, $F(x, y, z, w) = gx$. Similarly, we can show that $F(y, z, w, x) = gy$, $F(z, w, x, y) = gz$ and $F(w, x, y, z) = gw$.

Now, assume that $gx_0 \asymp gy_0 \asymp gz_0 \asymp gw_0$. From (2.6), we have

$$gx_n \asymp gx_0, \quad gy_n \asymp gy_0, \quad gz_n \asymp gz_0 \quad \text{and} \quad \mathrm{gw_n} \asymp \mathrm{gw_0}.$$

Then

$$\begin{aligned}&gx \asymp gx_n \asymp gx_0 \asymp gy_0 \asymp gy_n \asymp gy,\\ &gy \asymp gy_n \asymp gy_0 \asymp gz_0 \asymp gz_n \asymp gz\\ &\text{and} \quad gz \asymp gz_n \asymp gz_0 \asymp gw_0 \asymp gw_n \asymp gw\\ &\text{for } n \text{sufficiently large.}\end{aligned}$$

Hence $gx \asymp gy \asymp gz \asymp gw$. Therefor by (2.3) we obtain

$$\begin{aligned}&\psi(\max\{d(gx, gy), d(gy, gz), d(gz, gw), d(gw, gx)\})\\ &\quad \le \psi(s\max\{d(gx, gy), d(gy, gz), d(gz, gw), (gw, gx)\})\\ &\quad \le \psi(\max\{d(gx, gy), d(gy, gz), d(gz, gw),\\ &\quad d(gw, gx)\}) - \min\{\phi(d(gx, gy), d(gy, gz), d(gz, gw),\\ &\quad d(gw, gx)), \phi(d(gy, gz), d(gz, gw), d(gw, gx),\\ &\quad d(gx, gy)), \phi(d(gz, gw), d(gw, gx), d(gx, gy), d(gy, gz)),\\ &\quad\quad \phi(d(gw, gx), d(gx, gy), d(gy, gz), d(gz, gw))\}.\end{aligned}$$

Hence,

$$\begin{aligned}&\min\{\phi(d(gx, gy), d(gy, gz), d(gz, gw), d(gw, gx)),\\ &\phi(d(gy, gz), d(gz, gw), d(gw, gx), d(gx, gy)),\\ &\phi(d(gz, gw), d(gw, gx), d(gx, gy), d(gy, gz)),\\ &\phi(d(gw, gx), d(gx, gy), d(gy, gz), d(gz, gw))\} = 0,\end{aligned}$$

which implies that

$$\begin{aligned}&\phi(d(gx, gy), d(gy, gz), d(gz, gw), d(gw, gx))\\ &= 0 \text{ or } \phi(d(gy, gz), d(gz, gw), d(gw, gx), d(gx, gy))\\ &= 0 \text{ or } \phi(d(gz, gw), d(gw, gx), d(gx, gy), d(gy, gz))\\ &= 0 \text{ or } \phi(d(gw, gx), d(gx, gy), d(gy, gz), d(gz, gw))\} = 0.\end{aligned}$$

Then from the properties of ϕ we have $d(gx, gy) = d(gy, gz) = d(gz, gw) = d(gw, gx) = 0$, that is $gx = gy = gz = gw$. Now, suppose that $gx = gy = gz = gw = p$, since F and g are w-compatible, then

$$gp = ggx = g(F(x, y, z, w)) = F(gx, gy, gz, gw) = F(p, p, p, p).$$

So, F and g have a quadruple coincidence point of the form (p, p, p, p). □

Corollary 2.1 *Replace the contractive condition* (2.3) *of Theorem* 2.1 (*or Theorem* 2.2, *respectively*) *by the following condition:*

there exist $\psi : [0,\infty) \to [0,\infty)$ *such that* ψ *is an altering distance function and* $\varphi : [0,\infty) \to [0,\infty)$ *is continuous with* $\varphi(t) = 0$ *if and only if* $t = 0$ *such that*

$$\begin{aligned}&\psi(sd(F(x,y,z,w), F(u,v,r,t))) \le \psi(\max\{d(gx, gu),\\ &d(gy, gv), d(gz, gr), d(gw, gt)\})\\ &- \varphi(\max\{d(gx, gu), d(gy, gv), d(gz, gr), d(gw, gt)\}),\end{aligned}$$

for all $x, y, z, w, u, v, r, t \in X$ *and* $gx \asymp gu$, $gy \asymp gv$, $gz \asymp gr$ *and* $gw \asymp gt$. *Let the other conditions of Theorem* 2.1 (*or Theorem* 2.2, *respectively*) *be satisfied. Then,* F *and* g *have a quadruple coincidence point.*

Proof We replace $\phi(t_1, t_2, t_3, t_4) = \varphi(\max(t_1, t_2, t_3, t_4))$ in Theorem 2.1 (or Theorem 2.2, respectively). So φ is continuous and $\varphi(t) = 0$ if and only if $t = 0$. □

Corollary 2.2 *Replace the contractive condition (2.3) of Theorem 2.1 (or Theorem 2.2, respectively) by the following condition:*

$$d(F(x,y,z,w),F(u,v,r,t)) \leq \frac{k}{4s}\max\{d(gx,gu),$$
$$d(gy,gv),d(gz,gr),d(gw,gt)\},$$

for all $x,y,z,w,u,v,r,t \in X$, and $gx \asymp gu$, $gy \asymp gv$, $gz \asymp gr$ and $gw \asymp gt$, where $k \in [0,1)$. Let the other conditions of Theorem 2.1 (or Theorem 2.2) be satisfied. Then, F and g have a quadruple coincidence point.

Proof We take $\psi(t) = \frac{t}{4}$ and $\phi(t_1,t_2,t_3,t_4) = \frac{(1-k)}{4}\max(t_1,t_2,t_3,t_4)$ in Theorem 2.1 (or Theorem 2.2, respectively). □

Corollary 2.3 *Replace the contractive condition (2.3) of Theorem 2.1 (or Theorem 2.2, respectively) by the following condition:*

there exist $\phi : [0,\infty)^4 \to [0,\infty)$ is continuous with $\phi(t_1,t_2,t_3,t_4) = 0$ if and only if $t_1 = t_2 = t_3 = t_4 = 0$ such that

$$d(F(x,y,z,w),F(u,v,r,t)) \leq \frac{1}{s}\max\{d(gx,gu),d(gy,gv),$$
$$d(gz,gr),d(gw,gt)\} - \frac{1}{s}\phi(d(gx,gu),d(gy,gv),$$
$$d(gz,gr),d(gw,gt)),$$

for all $x,y,z,w,u,v,r,t \in X$ and $gx \asymp gu$, $gy \asymp gv$,$gz \asymp gr$ and $gw \asymp gt$. Let the other conditions of Theorem 2.1 (or Theorem 2.2, respectively) be satisfied. Then, F and g have a quadruple coincidence point.

Proof Taking $\psi(t) = t$ in Theorem 2.1 (or Theorem 2.2, respectively), we have Corollary 2.3. □

Corollary 2.4 *Replace the contractive condition (2.3) of Theorem 2.1 (or Theorem 2.2, respectively) by the following condition:*

there exist $\psi : [0,\infty) \to [0,\infty)$ such that ψ is an altering distance function, and $\phi : [0,\infty)^4 \to [0,\infty)$ is continuous with $\phi(t_1,t_2,t_3,t_4) = 0$ if and only if $t_1 = t_2 = t_3 = t_4 = 0$ such that

$$\psi(sd(F(x,y,z,w),F(u,v,r,t)))$$
$$\leq \psi\left(\frac{d(gx,gu)+d(gy,gv)+d(gz,gr)+d(gw,gt)}{4}\right)$$
$$- \phi(d(gx,gu),d(gy,gv),d(gz,gr),d(gw,gt)),$$

for all $x,y,z,w,u,v,r,t \in X$ and $gx \asymp gu$, $gy \asymp gv$, $gz \asymp gr$ and $gw \asymp gt$. Let the other conditions of Theorem 2.1 (or Theorem 2.2, respectively) be satisfied. Then, F and g have a quadruple coincidence point.

Proof Since

$$\frac{(d(gx,gu)+d(gy,gv)+d(gz,gr)+d(gw,gt)}{4}$$
$$\leq \max\{d(gx,gu),d(gy,gv),d(gz,gr),d(gw,gt)\},$$

and since ψ is assumed to be nondecreasing, then we apply Theorem 2.1 (or Theorem 2.2 respectively). □

Corollary 2.5 *Replace the contractive condition (2.3) of Theorem 2.1 (or Theorem 2.2 respectively) by the following condition*

$$d(F(x,y,z,w),F(u,v,r,t)) \leq \frac{k}{4s}(d(gx,gu)+d(gy,gv)$$
$$+d(gz,gr)+d(gw,gt)),$$

for all $x,y,z,w,u,v,r,t \in X$ and $gx \asymp gu$, $gy \asymp gv$,$gz \asymp gr$ and $gw \asymp gt$, where $k \in [0,1)$. Let the other conditions of Theorem 2.1 (or Theorem 2.2 respectively) be satisfied. Then F and g have a quadruple coincidence point.

Proof We take $\psi(t) = t$ and $\phi(t_1,t_2,t_3,t_4) = \left(\frac{1-k}{4}\right)(t_1+t_2+t_3+t_4)$ in Corollary 2.4. □

Now, we obtain some quadruple coincidence point results for mappings satisfying a contractive condition of integral type. We denote by Λ the set of all functions $\alpha : [0,+\infty) \to [0,+\infty)$ verifying the following conditions:

(i) α is a positive Lebesgue integrable mapping on each compact subset of $[0,+\infty)$

(ii) for all $\varepsilon > 0$, we have $\int_0^{\varepsilon}\alpha(t)dt > 0$.

Let $N \in \mathbf{N}$ be a fixed positive integer. Let $\{\alpha_i\}_{1\leq i\leq N}$ be a family of N functions that belong to Λ. For all $t \geq 0$, we denote $(I_i)_{i=1,\ldots,N}$ as follows:

$$I_1(t) = \int_0^t \alpha_1(s)\mathrm{d}s,$$
$$I_2(t) = \int_0^{I_1(t)} \alpha_2(s)\mathrm{d}s = \int_0^{\int_0^t \alpha_1(s)\mathrm{d}s} \alpha_2(s)\mathrm{d}s,$$
$$I_3(t) = \int_0^{I_2(t)} \alpha_3(s)\mathrm{d}s = \int_0^{\int_0^{\int_0^t \alpha_1(s)\mathrm{d}s} \alpha_2(s)\mathrm{d}s} \alpha_3(s)\mathrm{d}s,$$
$$\ldots$$
$$I_N(t) = \int_0^{I_{N-1}(t)} \alpha_N(s)\mathrm{d}s.$$

We have the following result.

Corollary 2.6 *Replace the contractive condition (2.3) of Theorem 2.1 (or Theorem 2.2 respectively) by the following condition:*

$$I_N(\psi(sd(F(x,y,z,w),F(u,v,r,t))))$$
$$\leq I_N(\psi(\max\{d(gx,gu),d(gy,gv),d(gz,gr),d(gw,gt)\}))$$
$$- I_N(\phi(d(gx,gu),d(gy,gv),$$
$$d(gz,gr),d(gw,gt))). \tag{2.27}$$

Let the other conditions of Theorem 2.1 (or Theorem 2.2 respectively) be satisfied. Then F and g have a quadruple coincidence point.

Proof Consider the function $\Psi = I_N \circ \psi$ and $\Phi = I_N \circ \phi$. Then (2.27) becomes

$$\begin{aligned}&\Psi(sd(F(x,y,z,w),F(u,v,r,t)))\\ &\le \Psi(\max\{d(gx,gu),d(gy,gv),d(gz,gr),d(gw,gt)\})\\ &\quad - \Phi(d(gx,gu),d(gy,gv),d(gz,gr),d(gw,gt)).\end{aligned}$$

It is easy to show that Ψ is an altering distance function, Φ is continuous and $\Phi(t_1,t_2,t_3,t_4)=0$ if and only if $t_1 = t_2 = t_3 = t_4 = 0$. Applying Theorem 2.1 (or Theorem 2.2 respectively) we obtain the proof. □

In the case $N = 1$, we have the following corollary.

Corollary 2.7 *Replace the contractive condition (2.3) of Theorems 2.1 (or Theorem 2.2 respectively) by the following: There exists $\alpha \in \Lambda$ such that*

$$\int_0^{\psi(sd(F(x,y,z,w),F(u,v,r,t)))} \alpha(t)\mathrm{d}t \le \int_0^{\psi(\max\{d(gx,gu),d(gy,gv),d(gz,gr),d(gw,gt)\})} \alpha(t)\mathrm{d}t - \int_0^{\phi(d(gx,gu),d(gy,gv),d(gz,gr),d(gw,gt))} \alpha(t)\mathrm{d}t.$$

Let the other conditions of Theorem 2.1 (or Theorem 2.2 respectively) be satisfied. Then F and g have a quadruple coincidence point.

Uniqueness of quadruple fixed point

In this section, we will show the uniqueness of a quadruple common fixed point.

For a product X^4 of a partially ordered set $(X,\preceq)$, we define a partial ordering in the following way. For all $(x,y,z,w),(u,v,r,t) \in X^4$,

$$(x,y,z,w) \preceq (u,v,r,t) \Leftrightarrow x \preceq u,\ y \succeq v,\ z \preceq r,\ w \succeq t. \tag{3.1}$$

We say that (x,y,z,w) and (u,v,r,t) are comparable if

$$(x,y,z,w) \preceq (u,v,r,t) \text{ or } (u,v,r,t) \succeq (x,y,z,w).$$

Also, we say that (x,y,z,w) is equal to (u,v,r,t) if and only if $x = u$, $y = v$, $z = r$, $w = t$.

Theorem 3.1 *In addition to hypotheses of Theorem 2.1 (or Theorem 2.2, respectively) assume that for all quadruple coincidence points (x,y,z,w), $(u,v,r,t) \in X^4$, there exists $(a,b,c,d) \in X^4$ such that $(F(a,b,c,d),F(b,c,d,a),F(c,d,a,b),F(d,a,b,c))$ is comparable to both $(F(x,y,z,w),F(y,z,w,x),F(z,w,x,y),F(w,x,y,z))$ and $(\mathrm{F(u,v,r,t),F(v,r,t,u),F(r,t,u,v),F(t,u,v,r)})$. Then F and g have a unique quadruple common fixed point (x,y,z,w) such that $x = gx = F(x,y,z,w)$, $y = gy = F(y,z,w,x)$, $z = gz = F(z,w,x,y)$, and $w = gw = F(w,x,y,z)$.*

Proof Theorem 2.1 (or Theorem 2.2 respectively) implies that The set of quadruple coincidence points of F and g is not empty. Suppose that (x,y,z,w) and (u,v,r,t) are two quadruple coincidence points of F and g, that is, $F(x,y,z,w) = gx$, $F(u,v,r,t) = gu$, $F(y,z,w,x) = gy$,$F(v,r,t,u) = gv$, $F(z,w,x,y) = gz$, $F(r,t,u,v) = gr$, $F(w,x,y,z) = gw$,$F(t,u,v,r) = gt$. We show that $(gx,gy,gz,gw) = (gu,gv,gr,gt)$. By assumption, there exists $(a,b,c,d) \in X^4$ such that $(F(a,b,c,d),F(b,c,d,a),F(c,d,a,b),F(d,a,b,c))$ is comparable to both

$$(F(x,y,z,w),F(y,z,w,x),F(z,w,x,y),F(w,x,y,z))$$
$$\text{and } (F(u,v,r,t),F(v,r,t,u),F(r,t,u,v),F(t,u,v,r)).$$

Since $F(X^4) \subseteq g(X)$, we can define the sequences $\{ga_n\},\{gb_n\},\{gc_n\}$ and $\{gd_n\}$ such that $a_0 = a$, $b_0 = b$, $c_0 = c$, $d_0 = d$, and

$$ga_{n+1} = F(a_n,b_n,c_n,d_n),\quad gb_{n+1} = F(b_n,c_n,d_n,a_n),$$
$$gc_{n+1} = F(c_n,d_n,a_n,b_n),\quad gd_{n+1} = F(d_n,a_n,b_n,c_n),$$

for all $n \ge 0$. Also, in the same way define the sequences $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$, $\{gw_n\}$ and $\{gu_n\}$,$\{gv_n\}$, $\{gr_n\}$, $\{gt_n\}$, such that $x_0 = x$, $y_0 = y$,$z_0 = z$, $w_0 = w$, and $u_0 = u$, $v_0 = v$, $r_0 = r$, $t_0 = t$, by

$$gx_{n+1} = F(x_n,y_n,z_n,w_n),\quad gy_{n+1} = F(y_n,z_n,w_n,x_n),$$
$$gz_{n+1} = F(z_n,w_n,x_n,y_n),\quad gw_{n+1} = F(w_n,x_n,y_n,z_n),$$

and

$$gu_{n+1} = F(u_n,v_n,r_n,t_n),\quad gv_{n+1} = F(v_n,r_n,t_n,u_n),$$
$$gr_{n+1} = F(r_n,t_n,u_n,v_n),\quad gt_{n+1} = F(t_n,u_n,v_n,r_n),$$

for all $n \ge 0$. Since (x,y,z,w) and (u,v,r,t) are quadruple coincidence points of F and g,, then $gx_n = F(x,y,z,w)$, $gu_n = F(u,v,r,t)$, $gy_n = F(y,z,w,x)$, $gv_n = F(v,r,t,u)$, $gz_n = F(z,w,x,y)$, $gr_n = F(r,t,u,v)$, $gw_n = F(w,x,y,z)$, $gt_n = F(t,u,v,r)$, for all $n \ge 0$.

Since $(F(x,y,z,w),F(y,z,w,x),F(z,w,x,y),F(w,x,y,z)) = (gx_1,gy_1,gz_1,gw_1) = (gx,gy,gz,gw)$ is comparable to $(F(a,b,c,d),F(b,c,d,a),F(c,d,a,b),F(d,a,b,c)) = (ga_1,gb_1,gc_1,gd_1)$, then it is easy to show $gx \asymp ga_1$, $gy \asymp gb_1$, $gz \asymp gc_1$, $gw \asymp gd_1$. In a similar way, we get that

$$gx \asymp ga_n,\quad gy \asymp gb_n,\quad gz \asymp gc_n,\quad gw \asymp gd_n \text{ for all } n. \tag{3.2}$$

From (2.3) and (3.2), we obtain

$$\begin{aligned}&\psi(sd(gx, ga_{n+1}))\\&= \psi(sd(F(x,y,z,w), F(a_n,b_n,c_n,d_n)))\\&\leq \psi(\max\{d(gx,ga_n), d(gy,gb_n), d(gz,gc_n), d(gw,gd_n)\})\\&\quad - \phi(d(gx,ga_n), d(gy,gb_n), d(gz,gc_n), d(gw,gd_n)),\end{aligned} \tag{3.3}$$

$$\begin{aligned}&\psi(sd(gy, gb_{n+1}))\\&= \psi(sd(F(y,z,w,x), F(b_n,c_n,d_n,a_n)))\\&\leq \psi(\max\{d(gy,gb_n), d(gz,gc_n), d(gw,gd_n), d(gx,ga_n)\})\\&\quad - \phi(d(gy,gb_n), d(gz,gc_n), d(gw,gd_n), d(gx,ga_n)),\end{aligned} \tag{3.4}$$

$$\begin{aligned}&\psi(sd(gz, gc_{n+1}))\\&= \psi(sd(F(z,w,x,y), F(c_n,d_n,a_n,b_n)))\\&\leq \psi(\max\{d(gz,gc_n), d(gw,gd_n), d(gx,ga_n), d(gy,gb_n)\})\\&\quad - \phi(d(gz,gc_n), d(gw,gd_n), d(gx,ga_n), d(gy,gb_n)),\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\psi(sd(gw, gd_{n+1}))\\&= \psi(sd(F(w,x,y,z), F(d_n,a_n,b_n,c_n)))\\&\leq \psi(\max\{d(gw,gd_n), d(gx,ga_n), d(gy,gb_n), d(gz,gc_n)\})\\&\quad - \phi(d(gw,gd_n), d(gx,ga_n), d(gy,gb_n), d(gz,gc_n)).\end{aligned} \tag{3.6}$$

Set

$$\gamma_n = \max\{d(gx,ga_n), d(gy,gb_n), d(gz,gc_n), d(gw,gd_n)\}.$$

By (3.3)–(3.6) and Remark 2.1, we obtain that

$$\begin{aligned}&\psi(\gamma_{n+1}) \leq \psi(s\gamma_{n+1})\\&\quad \leq \psi(\gamma_n) - \min\{\phi(d(gx,ga_n), d(gy,gb_n), d(gz,gc_n), d(gw,gd_n)),\\&\phi(d(gy,gb_n), d(gz,gc_n), d(gw,gd_n), d(gx,ga_n)),\\&\phi(d(gz,gc_n), d(gw,gd_n), d(gx,ga_n), d(gy,gb_n)),\\&\phi(d(gw,gd_n), d(gx,ga_n), d(gy,gb_n), d(gz,gc_n))\} \leq \psi(\gamma_n).\end{aligned} \tag{3.7}$$

Hence,

$$\psi(\gamma_{n+1}) \leq \psi(\gamma_n) \qquad \text{for all } n \in \mathbf{N}.$$

Since ψ is nondecreasing, then $\gamma_{n+1} \leq \gamma_n$ for all n. This implies that γ_n is a non-increasing sequence. Therefore, there exists $\gamma \geq 0$ such that
$\lim_{n\to\infty} \gamma_n = \gamma.$

We show that $\gamma = 0$. Letting $n \to \infty$, in (3.7), we get

$$\begin{aligned}&\psi(\gamma) \leq \psi(\gamma) - \min\{\lim_{n\to\infty} \phi(d(gx,ga_n), d(gy,gb_n),\\&d(gz,gc_n), d(gw,gd_n)),\\&\lim_{n\to\infty} \phi(d(gy,gb_n), d(gz,gc_n), d(gw,gd_n), d(gx,ga_n)),\\&\quad \lim_{n\to\infty} \phi(d(gz,gc_n), d(gw,gd_n), d(gx,ga_n), d(gy,gb_n)),\\&\quad \lim_{n\to\infty} \phi(d(gw,gd_n), d(gx,ga_n), d(gy,gb_n), d(gz,gc_n))\} \leq \psi(\gamma).\end{aligned}$$

Hence,

$$\begin{aligned}&\min\{\lim_{n\to\infty} \phi(d(gx,ga_n), d(gy,gb_n), d(gz,gc_n), d(gw,gd_n)),\\&\quad \lim_{n\to\infty} \phi(d(gy,gb_n), d(gz,gc_n), d(gw,gd_n), d(gx,ga_n)),\\&\quad \lim_{n\to\infty} \phi(d(gz,gc_n), d(gw,gd_n), d(gx,ga_n), d(gy,gb_n)),\\&\lim_{n\to\infty} \phi(d(gw,gd_n), d(gx,ga_n), d(gy,gb_n), d(gz,gc_n))\} = 0.\end{aligned}$$

Therefore,

$$\begin{aligned}&\lim_{n\to\infty} \phi(d(gx,ga_n), d(gy,gb_n), d(gz,gc_n), d(gw,gd_n)) = 0 \quad \text{or}\\&\lim_{n\to\infty} \phi(d(gy,gb_n), d(gz,gc_n), d(gw,gd_n), d(gx,ga_n)) = 0 \quad \text{or}\\&\lim_{n\to\infty} \phi(d(gz,gc_n), d(gw,gd_n), d(gx,ga_n), d(gy,gb_n)) = 0 \quad \text{or}\\&\quad \lim_{n\to\infty} \phi(d(gw,gd_n), d(gx,ga_n), d(gy,gb_n), d(gz,gc_n))\} = 0.\end{aligned}$$

Using the properties of ϕ, we have

$$\begin{aligned}&\lim_{n\to\infty} d(gx,ga_n) = \lim_{n\to\infty} d(gy,gb_n) = \lim_{n\to\infty} d(gz,gc_n)\\&\quad = \lim_{n\to\infty} d(gw,gd_n) = 0.\end{aligned} \tag{3.8}$$

Thus $\lim_{n\to\infty} \gamma_n = 0$. Similarly, we can show that

$$\begin{aligned}&\lim_{n\to\infty} d(gu,ga_n) = \lim_{n\to\infty} d(gv,gb_n)) = \lim_{n\to\infty} d(gr,gc_n)\\&\quad = \lim_{n\to\infty} d(gt,gd_n) = 0.\end{aligned} \tag{3.9}$$

From (3.8) and (3.9), we conclude that $(gx,gy,gz,gw) = (gu,gv,gr,gt)$, That is the quadruple coincidence point of F and g is unique.

Denote $gx = x^*$, $gy = y^*$, $gz = z^*$, $gw = w^*$ and since $gx = F(x,y,z,w)$, $gy = F(y,z,w,x)$, $gz = F(z,w,x,y)$, $gw = F(w,x,y,z)$ so we have that

$$gx^* = ggx = gF(x,y,z,w), \qquad gy^* = ggy = gF(y,z,w,x), \tag{3.10}$$

$$gz^* = ggz = gF(z,w,x,y) \text{ and } gw^* = ggw = gF(w,x,y,z). \tag{3.11}$$

By definition of the sequences $\{gx_n\}, \{gy_n\}, \{gz_n\}$ and $\{gw_n\}$, we have

$$gx_n = F(x, y, z, w) = F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}),$$
$$gy_n = F(y, z, w, x) = F(y_{n-1}, z_{n-1}, w_{n-1}, x_{n-1}),$$
$$gz_n = F(z, w, x, y) = F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}),$$
$$gw_n = F(w, x, y, z) = F(w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}).$$

Consequently,

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} F(x_n, y_n, z_n, w_n) = F(x, y, z, w), \tag{3.12}$$
$$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} F(y_n, z_n, w_n, x_n) = F(y, z, w, x), \tag{3.13}$$
$$\lim_{n\to\infty} gz_n = \lim_{n\to\infty} F(z_n, w_n, x_n, y_n) = F(z, w, x, y), \tag{3.14}$$
$$\lim_{n\to\infty} gw_n = \lim_{n\to\infty} F(w_n, x_n, y_n, z_n) = F(w, x, y, z). \tag{3.15}$$

Case 1: In Theorem 2.1, from compatibility and continuity of F and g we obtain

$$\begin{aligned}
&\lim_{n\to\infty} d(gF(x_n, y_n, z_n, w_n), F(gx_n, gy_n, gz_n, gw_n)) = 0,\\
&\lim_{n\to\infty} d(gF(y_n, z_n, w_n, x_n), F(gy_n, gz_n, gw_n, gx_n)) = 0,\\
&\lim_{n\to\infty} d(gF(z_n, w_n, x_n, y_n), F(gz_n, gw_n, gx_n, gy_n)) = 0,\\
&\lim_{n\to\infty} d(gF(w_n, x_n, y_n, z_n), F(gw_n, gx_n, gy_n, gz_n))) = 0,
\end{aligned} \tag{3.16}$$

where

$$\lim_{n\to\infty} gF(x_n, y_n, z_n, w_n) = gF(x, y, z, w),$$
$$\lim_{n\to\infty} F(gx_n, gy_n, gz_n, gw_n) = F(gx, gy, gz, gw)),$$
$$\lim_{n\to\infty} gF(y_n, z_n, w_n, x_n) = gF(y, z, w, x),$$
$$\lim_{n\to\infty} F(gy_n, gz_n, gw_n, gx_n) = F(gy, gz, gw, gx),$$
$$\lim_{n\to\infty} gF(z_n, w_n, x_n, y_n) = gF(z, w, x, y),$$
$$\lim_{n\to\infty} F(gz_n, gw_n, gx_n, gy_n) = F(gz, gw, gx, gy),$$
$$\lim_{n\to\infty} gF(w_n, x_n, y_n, z_n) = gF(w, x, y, z),$$
$$\lim_{n\to\infty} F(gw_n, gx_n, gy_n, gz_n)) = F(gw, gx, gy, gz).$$

Thus from Lemma 1.1, we conclude that

$$gF(x, y, z, w) = F(gx, gy, gz, gw), \quad gF(y, z, w, x) = F(gy, gz, gw, gx),$$
$$gF(z, w, x, y) = F(gz, gw, gx, gy), \quad gF(w, x, y, z) = F(gw, gx, gy, gz).$$

Moreover, from (3.10) implies that

$$gx^* = F(x^*, y^*, z^*, w^*),\ gy^* = F(y^*, z^*, w^*, x^*),$$
$$gz^* = F(z^*, w^*, x^*, y^*),\ gw^* = F(w^*, x^*, y^*, z^*).$$

Case 2: In Theorem 2.2 since F and g are w-compatible, then

$$gx^* = ggx = g(F(x, y, z, w)) = F(gx, gy, gz, gw) = F(x^*, y^*, z^*, w^*)$$
$$gy^* = ggy = g(F(y, z, w, x)) = F(gy, gz, gw, gx) = F(y^*, z^*, w^*, x^*)$$
$$gz^* = ggz = g(F(z, w, x, y)) = F(gz, gw, gx, gy) = F(z^*, w^*, x^*, y^*)$$
$$gw^* = ggw = g(F(w, x, y, z)) = F(gw, gx, gy, gz) = F(w^*, x^*, y^*, z^*).$$

Thus, in the two cases we conclude that (x^*, y^*, z^*, w^*) is another quadruple coincidence point of F and g. Hence, $(gx^*, gy^*, gz^*, gw^*) = (gx, gy, gz, gw)$. Therefore

$$gx^* = gx = x^*, \quad gy^* = gy = y^*, \quad gz^* = gz = z^*,$$
$$\text{and} \quad gw^* = gw = w^*.$$

Hence, (x^*, y^*, z^*, w^*) is a quadruple common fixed point of F and g. The uniqueness of a quadruple common fixed point follows easily from the uniqueness of a quadruple coincidence point. □

Now, we give an example to justify the hypotheses of Theorem 2.1.

Example 3.1 Let $X = [0, \infty)$ be equipped with the b-metric $d(x, y) = (x - y)^2$ for all $x, y \in X$, where $s = 2$, and suppose that $\preceq$ is the usual ordering $\leq$ on X. Obviously, $(X, d, \preceq)$ is a partially ordered complete b-metric space. Let $F : X^4 \to X$ and $g : X \to X$ be defined by

$$F(x, y, z, w) = \frac{x^2 + y^2 + z^2 + w^2}{16} \quad \textit{and} \quad g(x) = x^2.$$

It is easy to see that g and F are compatible. Define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = kt$ and $\phi : [0, \infty)^4 \to [0, \infty)$ by $\phi(t_1, t_2, t_3, t_4) = \frac{k-1}{4}(t_1 + t_2 + t_3 + t_4)$, where $1 \leq k \leq 8$. Then, ψ and ϕ have the properties mentioned in Theorem 2.1. Further, for all $x, y, z, w, u, v, r, t \in X$, we have $gx \asymp gu$, $gy \asymp gv$, $gz \asymp gr$ and $gw \asymp gt$. Hence,

$$\begin{aligned}
&\psi(sd(F(x, y, z, w), F(u, v, r, t)))\\
&= \psi\left(2\left(\frac{x^2 + y^2 + z^2 + w^2}{16} - \frac{u^2 + v^2 + r^2 + t^2}{16}\right)^2\right)\\
&= \frac{k}{128}((x^2 - u^2) + (y^2 - v^2) + (z^2 - r^2) + (w^2 - t^2))^2\\
&\leq \frac{4k}{128}\left((x^2 - u^2)^2 + (y^2 - v^2)^2 + (z^2 - r^2)^2 + (w^2 - t^2)^2\right)\\
&= \frac{k}{32}(d(gx, gu) + d(gy, gv) + d(gz, gr) + d(gw, gt))\\
&\leq \frac{8}{32}(d(gx, gu) + d(gy, gv) + d(gz, gr) + d(gw, gt))\\
&= \frac{1}{4}(d(gx, gu) + d(gy, gv) + d(gz, gr) + d(gw, gt))\\
&= \frac{k}{4}(d(gx, gu) + d(gy, gv) + d(gz, gr) + d(gw, gt))\\
&\quad - \frac{k-1}{4}(d(gx, gu) + d(gy, gv) + d(gz, gr) + d(gw, gt))\\
&\leq k \max\{d(gx, gu), d(gy, gv), d(gz, gr), d(gw, gt)\}\\
&\quad - \phi(d(gx, gu), d(gy, gv), d(gz, gr), d(gw, gt))\\
&= \psi(\max\{d(gx, gu), d(gy, gv), d(gz, gr), d(gw, gt)\})\\
&\quad - \phi(d(gx, gu), d(gy, gv), d(gz, gr), d(gw, gt)).
\end{aligned}$$

So, F and g satisfy all the conditions of Theorem and $(0,0,0,0)$ is a quadruple coincidence point of F and g. Moreover, by Theorem 3.1 $(0,0,0,0)$ is the unique quadruple common fixed point of F and g.

Note that, in this case F does not have the g-mixed monotone property, so the results of paper [25] cannot be applied.

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